<u>Derangements* and Random Rearrangements: An Exploration of Probability</u>

*A derangement in mathematics is a rearrangement of a set such that no element of the set is in its original position

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This lesson plan is based on material provided by Brian Conrey and Tom Davis during the American Institute of Mathematics (AIM) Math Teachers' Circle workshop in July 2016 (www.geometer.org/mathcircles).

Introduction

In middle school mathematics instruction, probability can sometimes be relegated to relatively simple problems about choosing colored marbles from a paper sack, or tossing a pair of dice. While these activities develop a basic understanding of necessary concepts, probability is much more intriguing than most students are aware! Probability can be counterintuitive. While we *feel* that a certain probability should be correct, the reality may be quite different. Even more fascinating, experimental probability generally mirrors theoretical probability quite neatly, even when it doesn't *feel* right.

In this lesson, students will develop understanding of both experimental and theoretical probability through a high-interest, story-based scenario with an unexpected twist. They will also investigate patterns, make predictions, and identify transferable problem-solving strategies.

This lesson is intended for seventh or eighth grade students and is based on the seventh grade Common Core Standards. It has been field-tested with both sixth and seventh grade level students, as well as more advanced students (eighth grade and above) and it contains high-level concepts that make it easily extendable for older or more advanced students.

Learning Objectives

Standards for grade seven in the "Statistics and Probability" domain, especially:

- 7.SP.C.5 Understand that the probability of a chance event is a number between 0 and 1 (prerequisite knowledge that is reinforced in this lesson)
- 7.SP.C.6 Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency; predict the approximate relative frequency given the probability
- 7.SP.C.7 Develop a probability model and use it to find probabilities of events; compare probabilities from a model to observed frequencies

Additionally, students will develop their mathematical practice skills in all eight standards, with special focus on the following Common Core Math Practice Standards:

- MP1: Make sense of problems and persevere in solving them
- MP4: Model with mathematics
- MP7: Look for and make use of structure

While not a Common Core standard at this level, students also work with combinatorics concepts during the lesson, a discrete math skill that may be explored in more depth in high school.

Prerequisite Knowledge

Ideally, students should have a basic understanding of probability before attempting the activities in this lesson. If students do not already recognize, for example, that drawing a red marble out of a bag with three marbles, one of which is red, has a probability of one out of three or 1/3, or that a certain event has a probability of 1 and an impossible event a probability of 0, this lesson may be difficult for them. However, some review is built into the early stages of the lesson and this could be extended to provide additional support. Students also need to know how to convert between fractions and decimals using division, or should have access to a calculator for this process.

Time Required

Preparation Time: Approximately 15 minutes

Class Time: Two 50-minute class periods

Materials

- One deck of cards for every four students
- A numbered “ticket” for each seat in the classroom
- Chart paper, butcher paper, or whiteboard to record data
- Theoretical probability data collection sheets (optional)
- Experimental data recording sheets (optional)
- Masking tape

Set Up

In advance, identify each seat in the classroom with a number. Masking tape works well to mark the seats. Create a small ticket with a matching number for each seat in the classroom. Place these numbers in a basket or bowl. This will be used for the hook activity.

Also, for each deck of cards, remove the jokers and split the deck in half, so that each pair of students will receive two suits from the deck for the data-gathering exploration.

If desired, make copies of the student data collection sheets for each pair of students. These can be used to facilitate this exploration, but may be eliminated if teachers wish to focus on data organization as a skill during this lesson. If this is the case, students may be asked to generate ideas for how to organize their data and, with guidance from the teacher, may create their own data collection sheets using graph or notebook paper.

Lesson Hook [3-5 minutes, ideally repeated over several days]

In advance of this lesson, students should identify or be assigned a “seat number” for their usual classroom seat using numbers or a letter-number combination. As students enter the classroom on the day of the lesson, provide them with a randomly selected “ticket” to a particular seat where they will sit during the lesson. Once all students have arrived, discuss: Were any students randomly assigned to their regular classroom seats? Is this unexpected? How likely do you think it is that this would happen? What if our class were much bigger? What if it were much smaller? Would it be more likely, less likely, or equally likely that at least one student seated randomly would end up in his or her own seat?

If possible, conduct this experiment during the first few minutes of class for several days leading up to this lesson to prime students for the ideas they will be exploring and to build student interest and curiosity.

Introducing the Problem: A Trip to Yankee Stadium [5-10 minutes]

Explain to students: *“Let’s consider a similar situation with* ***many*** *more people. It is the final game of the world series at Yankee stadium, and the game is completely sold out. Spectators arrive outside of the stadium holding their tickets, each with a designated row and seat number. Before they enter, however, they receive an unexpected announcement. All ticketed spectators will be seated at random inside the stadium. If the 54,251 spectators are seated randomly in the 54,251 seats in Yankee stadium, what is the probability that at least one of them will be seated in the seat indicated on their ticket?”*

Remind students that the maximum value for the probability of an event is 1, which means that the event is completely certain. The minimum value for a probability is 0, which means that the event is impossible. Events that are quite likely have probability values close to 1, while those that are quite unlikely have values close to 0. If students are struggle with these concepts of basic probability, this review may be extended.

Ask students to think about the probability value they’d assign to the event above, that at least one person in Yankee stadium would be randomly seated in his or her own ticketed seat. They should also think about a justification for their answer. After a brief thinking period, invite students to share their thoughts with students seated near them in pairs or small groups, and then select a few to share their answers with the whole group. As a class, consider: Do all students have similar answers? Do these answers seem reasonable? Are there any outliers in student opinions? What justifications do students give for their ideas?

Exploration #1: Experiment with Playing Cards [20 minutes]

Explain to students: *"One way to attack this problem would be to travel to Yankee stadium, conduct this experiment, and then tally results. We could then repeat the experiment many times and see if our answers suggest a likely probability. However, 54,251 seats is an awfully large number to work with, and the trip is probably not in the budget. A common problem-solving strategy for mathematicians is to start with a simpler problem and collect some data, so let's start with a much smaller 13-seat "stadium" and experiment."*

Give each student or pair of students two suits from a deck of playing cards (such as spades and hearts). Have them lay one suit out in front of them, in order, like this:

A♠ 2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠

Then, shuffle the second suit to randomize the cards and deal them below those already laid out, like this:

A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
7♡	J♡	A♡	2♡	3♡	9♡	K♡	4♡	8♡	10♡	2♡	5♡	6♡

For this particular deal, the 10 of hearts lands under the 10 of spades, so this is a match. In stadium language, the person with a 10-ticket landed in the 10-seat. But is this an expected result, or just an unusually lucky deal? We need more data to find out.

Students should repeat this experiment multiple times, tallying the number of times that there is a match as well as the number of times that they repeated the experiment. Then, together, tally the results for the entire class. With students, calculate the experimental probability for this event given the class data. This is a good time to review how to calculate a probability if students need this. Highlight the fact that, because we gathered our data by conducting an experiment in the real world, our probability is called "experimental probability."

For example, you might explain to students: "*Determining probability through experiments in the real world is called **experimental probability**. Experimental probability is recorded as a fraction where the denominator is the **total number of trials** and the numerator is the **number of desired outcomes.** As a class, we conducted a total of 85 trials, so this is our denominator. Our "desired outcome" was at least one match, and the total times that we got at least one match in our class was 65, so our experimental probability was 65/85 or 13/18."*

As a class or in small groups or pairs, briefly discuss: Given our class experimental data, in a 13-seat stadium, what is the approximate probability that at least one of the spectators will sit in their own seat? Did you expect these results? Why or why not? Given this information, would you like to revise your initial hypothesis about Yankee stadium?

Introducing Theoretical Probability / Wrap-up Day One [5-10 minutes]

At this point, you may wish to conclude the lesson by giving a brief demonstration of a coin flip. Students generally can easily understand that, because there are two possibilities, heads and tails, we'd expect that flipping a coin would give heads about half the time. You may try this in class and record the data on the board, of you could skip the demonstration and move directly to the closing explanation.

Explain to students: *"While experiments can give us an idea of the probability value that we might expect, there is variation in experimental data. Looking at our class data, some student pairs had very different data from other student pairs (highlight some examples). When we are working with experimental probability, there is always variation. For example, if I flip a coin ten times, I'd expect that I'd get heads about half the time, but I might get heads six or seven times, or three or four times. There is variation, even if our coin is perfectly fair. Tomorrow, when you come to math class, we will be working with* ***theoretical probability****. This means that we will be figuring out the probability that we would expect to happen, using math, rather than trying it out in the real world. With the coin flip, I'd expect to get heads half the time - that's my theoretical probability. But I might actually get heads seven times - that's my* ***experimental probability****. We'll work more with this difference tomorrow."*

Exit Ticket [2-3 minutes]

As an exit ticket, I asked students to record a conjecture about the theoretical probability for at least one match in Yankee Stadium on a sheet of paper, notecard, or Post-it note and hand it to me before leaving class. It could be the same as the one they developed at the beginning of class, or it could be revised based on our experiment.

Introduction Day Two [3-5 minutes]

Explain to students: *"Remember that yesterday we started with a question - if 54,251 people were seated in the 54,251 seats in Yankee Stadium, and then all were randomly assigned new seats, what is the probability that at least one spectator would be seated in his or her own seat? We conducted experiments to help us think about this problem, but we recognized that there is variation when we do real-world experiments, so today we are going to try to determine the theoretical probability, or what we'd expect the probability to be, using math. To calculate theoretical probability, we again use a fraction, where the denominator is* ***all the possible outcomes****, and the numerator is* ***all the possible desired outcomes****. For example, in a coin flip, there are two possible outcomes - heads and tails - and only one of them is our desired outcome. If I call heads, then heads is my desired outcome. So my theoretical probability of getting heads is 1/2."*

Exploration #2: Determining probability with small sets [30 minutes, or 40-45 minutes if Exploration #3 is built into periodic check-ins during Exploration #2]

Explain to students: "*54,251 seats is still much too big to work with, but we are going to do what mathematicians do and simplify the problem. Can we calculate the theoretical probability, at least for some very small stadium sizes?"*

Hand students the Experimental Data Recording Sheets. With students, consider a stadium with only one seat. What is the probability that a ticket-holder will randomly be assigned to the seat indicated on his or her ticket? Since there is only one seat and one ticket, it is absolutely certain that this will happen, so the probability is 1, as you can see in a one-card set:

1st Arrangement (Match)
A♠
A♡

What about a two-seat stadium? This is slightly more challenging than a one-seat stadium, but not much. To determine the probability, we need to know how many total possible arrangements there are and how many of those arrangements result in at least one match, which we can certainly figure out for just two seats! There are only two possible arrangements, and one of the two arrangements results in a match, so the probability of a match is ½ or 0.5.

1st Arrangement (Match)
A♠ 2♠
A♡ 2♡

2nd Arrangement (Derangement)
A♠ 2♠
2♡ A♡

Students ideally will be prepared at this point to continue this exploration on their own, but if not you can work through an additional example. In a three-seat stadium, there are six arrangements, four of which include at least one match:

1st Arrangement (Match)
A♠ 2♠ 3♠
A♡ 2♡ 3♡

2nd Arrangement (Match)
A♠ 2♠ 3♠
A♡ 3♡ 2♡

3rd Arrangement (Match)
A♠ 2♠ 3♠
2♡ A♡ 3♡

4th Arrangement (Derangement)
A♠ 2♠ 3♠
2♡ 3♡ A♡

5th Arrangement (Derangement)
A♠ 2♠ 3♠
3♡ A♡ 2♡

6th Arrangement (Match)
A♠ 2♠ 3♠
3♡ 2♡ A♡

Provide students with time to continue this exploration independently, determining the probability for at least one match in a four-seat stadium. After they have determined the probability for at least one match in a four-seat stadium, pause and invite students to share their methods for recording their data. What methods for recording made it easier to see all of the possible arrangements? Highlight the importance of organizing data as a problem-solving strategy. As a class, identify one or more ways of recording arrangements that will make it possible to tackle this problem for a five-seat stadium. Students may have identified the pattern for possible arrangements by this time, and may recognize that there are 120 possible arrangements for a five-seat stadium. This is very difficult for one person to write out! However, if pairs of students each take a portion of the total arrangements to write out, it becomes a more possible task. If one pair of students writes all the arrangements that begin with an Ace (or a 1), another pair all the arrangements that begin with a 2, a third all the arrangements that begin with a 3, and so on, each pair only needs to write out 24 possible arrangements. This is the same as for a four-seat stadium, and is doable. Students may even identify patterns that will simplify this process further. After supporting students in determining a method for finding data for a 5-seat stadium, provide time for them to work on this problem.

Their final data should look something like this:

Seats	**Possible Arrangements**	**Matches**	**Probability of at least one match (fraction)**	**Probability of at least one match (decimal)**
1	1	1	1/1	1
2	2	1	$\frac{1}{2}$	0.5
3	6	4	$4/6 = \frac{2}{3}$	0.666...
4	24	15	$15/24 = \frac{5}{8}$	0.625
5	120	76	76/120 = 19/30	0.633...

Exploration #3: Noticing Patterns and Making Predictions [10-15 minutes, or this process may occur at periodic check-in breaks during Exploration #2]

Explain to students: *"While we can calculate theoretical probability for very small mini-stadiums, working out all the possibilities is clearly not reasonable for Yankee stadium. Let's look more closely at the information that we have so far. Can we identify any patterns that would at least let us approximate the probability that we are looking for in Yankee stadium?"*

Allow students a discussion period to consider the data, notice patterns, make predictions, and share questions. As you circulate through the class, identify insightful noticings, hypotheses, predictions, and questions and invite students to share with the whole group. Students may benefit from the following guiding questions to provide support for their observations:

1. What do you notice about the number of possible arrangements for each number of seats? How is this changing? Can you predict the number of possible arrangements for stadiums with more than five seats using the pattern that you observe?

    The number of possible arrangements is the factorial for the number of seats, and students are likely to notice this pattern even if they do not yet know this vocabulary word. As students observe this, they may wonder why. This could lead to a discussion of why combinations can be found through factorials (there are N different possible options for the first seat, but then there are N-1 possibilities for the second seat, because one is already taken, and N-2 for the third seat because two seats are already taken, and so on; if these numbers are multiplied together, you get all the possible arrangements, just like finding all possible arrangements of outfits by multiplying the number of shirt choices and the number of pants choices).

2. When you look at the probability that there will be at least one match in a stadium, what do you notice about the decimal representations of the probability? How are these numbers changing? Can you make a prediction about a six-seat stadium? What do you think would be the highest possible probability (upper bound) of at least one match in a six-seat stadium? What do you think would be the lowest possible probability (lower bound)?

    The probabilities go up and down: $1 > 0.5 < 0.66\ldots > 0.625 < 0.633\ldots$ These values oscillate around a point that they are converging on quite quickly. This is easier to see in the decimal expansions of the fractional probability. You may wish to highlight that changing the representation of a number can give us different information, and this can be useful as a problem-solving strategy.

3. Find the differences between the fractional representations for the probability of at least one match ($1 - \frac{1}{2}$, $\frac{1}{2} - \frac{2}{3}$, and so on). What do you notice about the pattern of these differences. Can you use this information to predict the probability of at least one match in stadiums with more than five seats using the pattern that you observe?

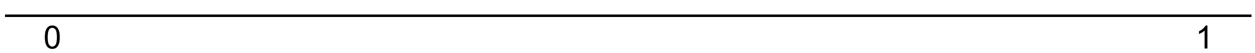
    If they do the fraction arithmetic, students can determine the difference between each successive probability and discover that there is also a pattern in these differences, and that it is related to the pattern in the number of possible arrangements (column two in the table above):
    $1-\frac{1}{2} = \frac{1}{2}$
    $\frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$
    $\frac{2}{3} - \frac{5}{8} = 1/24$
    $\frac{5}{8} - 19/30 = -1/120$
    Because the probability value goes up and down, the sign before the differences changes from positive to negative and back to positive again in a very predictable way. You may wish to highlight that doing this arithmetic with fractions makes this pattern clear, while doing the arithmetic with the decimal expansions of these fractions would not. Again, changing the representation of a number can be useful as a problem-solving strategy.

Students may also be interested in a pattern in the number of matches, but this is a significantly more challenging problem. Advanced students may be interested in working on this as an extension. More information about this is found under the "Extensions" portion of this lesson.

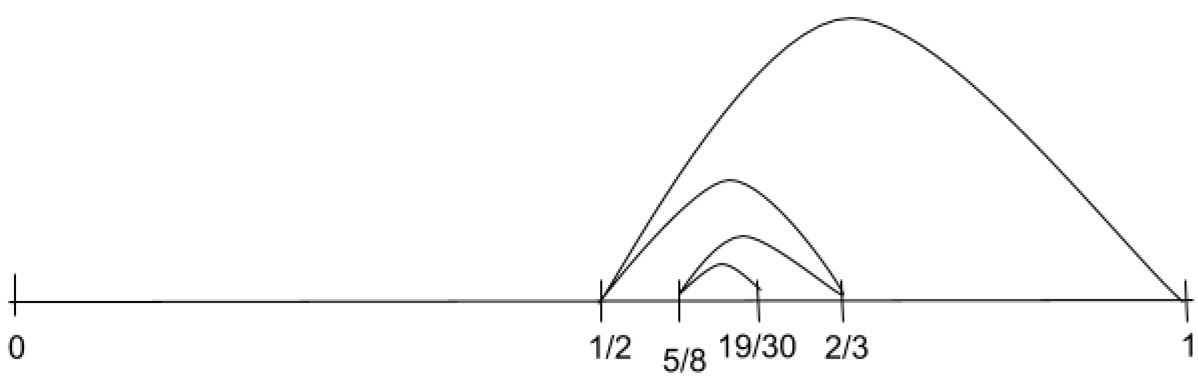
Debrief/Wrap-up [5-10 minutes]

Bring the class together to debrief. Review the original problem: What is the probability, in Yankee stadium, that at least one of the 54,251 spectators will be randomly seated in the seat indicated on their ticket? The discussions and explorations in this lesson will likely lead to some reasonable predictions. We can at least determine upper and lower bounds for this number, which will give a solution that is quite close to the theoretical probability, or the probability that we'd expect.

It is easiest to demonstrate the solution to this problem using a number line drawn on the board that looks like this:



Remind students that all probabilities fall between 0 and 1, with a probability of 0 being impossible, and a probability of 1 being certain. Mark the probability of a match in a 1-seat stadium at 1, then a probability of a match in a 2-seat stadium at ½, and connect these two positions on the number using an arc above the number line. Continue with the probability in a 3-seat stadium at ⅔, a 4-seat stadium at ⅝, and a 5-seat stadium at 19/30. Your finished number line will look something like this:



It is possible that using decimals instead of fractions may make these numbers easier to place on the number line.

At this point, students should notice something remarkable - the numbers are converging on a single value! This value is our probability, even for very, very large stadiums. Because of how quickly the probability is converging, a 13-seat stadium or even a 5-seat stadium does not yield a significantly different probability than a

54,251-seat stadium. This idea is likely to challenge students' intuitive expectations that these different situations should yield significantly different answers! Students are likely to want to know the *exact* answer, or the number that this pattern is converging on. This number is irrational (this might be a time to quickly review or discuss the concept of irrational numbers, if desired), but it can be rounded to approximately 0.632. The exact answer, which is unlikely to mean much to students at this point but can be entered into a calculator, is $1-1/e$.

Close this lesson by documenting problem-solving strategies that were important during this lesson (such as ways of organizing data, or the idea of trying an experiment and making a prediction, or the important strategy of starting with a simpler problem and working up, looking for patterns) and by sharing new mathematical understanding. This could include a deeper understanding of experimental and theoretical probability, and will almost certainly include the understanding that what *feels* right in probability does not necessarily prove true mathematically or in real-world experiments. This is an important idea. As students develop increased awareness that our intuitive sense of probability does not necessarily hold up under mathematical scrutiny, they will develop important mathematical thinking skills.

As students leave, ask them to complete the Exit Ticket to assess their understanding of basic experimental and theoretical probability.

Experimental Data Recording Sheet

Trial Number:	Match?	Derangement?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
Total		

Theoretical Probability Data Sheet

Seats	Possible Arrangements	Matches	Probability of at least one match (fraction)	Probability of at least one match (decimal)
1				
2				
3				
4				
5				

Exit Ticket Day Two

Exit Ticket

If you flip a fair coin, what is the probability that you will get heads?

Is this **experimental** or **theoretical** probability?

If you flip a coin 10 times, and get heads for 7 of them, what was your probability of getting heads?

Is this **experimental** or **theoretical** probability?

Exit Ticket

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Is this **experimental** or **theoretical** probability?

If you flip a coin 10 times, and get heads for 7 of them, what was your probability of getting heads?

Is this **experimental** or **theoretical** probability?

Extensions

There are many avenues to extend the ideas in this lesson for more advanced students. For example:

1. Comparing Theoretical and Experimental Probability: Once students have identified patterns in Exploration #3 that allow them to predict probabilities for stadiums larger than five, can they determine the theoretical probability for a 13-seat stadium? How does this compare with the class data for the 13-seat stadium experiment with playing cards?

2. Noticing More Patterns: In Exploration #3, students observed patterns in their data and made predictions. Finding a pattern in the number of matches in a set of all possible arrangements (column 3 in the table above) is a more challenging problem, but some students may be interested in exploring it. They may notice patterns in their data that allow them to make extrapolations and conjectures about the continuation of this pattern. They might also be able to work backwards from the probability pattern to determine the next numbers in this sequence, which could give more data to work with.

3. "Oscillating" Numbers: The probability of at least one match in a randomly seated stadium converges toward a single value quite quickly, but it does this by bouncing above and below the value that it is approaching (1 - 1/*e*). Other sequences have a similar property. For example, the ratios of successive numbers in the Fibonacci sequence oscillate before converging on the value phi. Students may wish to investigate this sequence, as well as research others that involve this interesting phenomenon.

As mentioned earlier, some very advanced students may be interested in examining the pattern in the number of matches. These students might be presented with the following information, which would challenge high school students:

Determining a formula for the number of possible matches

Determining a formula to find the exact number of derangements of a set, and by extension the number of possible "matches" in any given set (column 3 in the table under "Figuring out probability with small sets") is a more challenging problem, but it is possible. Advanced students may be interested in trying to find a formula for these numbers. They could begin by trying to model with mathematics the data that they are seeing, and may come up with some insightful observations that will help them to approximate a general formula for these values.

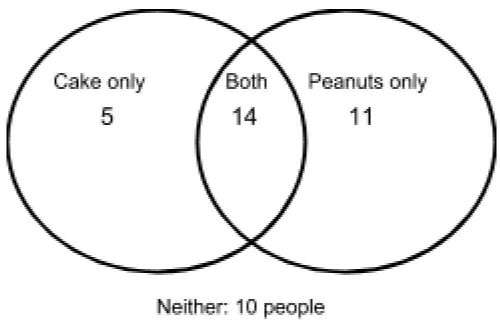
Ultimately, for students to understand how to compute these values, they may need to understand two additional mathematical ideas: the inclusion-exclusion principle for combinatorics and how to determine the number of possible combinations of k members in a set of size N (N choose k formula).

Inclusion-Exclusion Principle

To begin to understand this principle, consider a different, but related, problem. Imagine that you have a group of 40 people, and you ask each of them about their feelings about cake and peanuts. Nineteen say that they like cake, while 25 tell you they like peanuts. You want to find out how many people like neither cake nor peanuts, so you subtract the 40 - 19 - 25 = -4 people like neither cake nor peanuts.

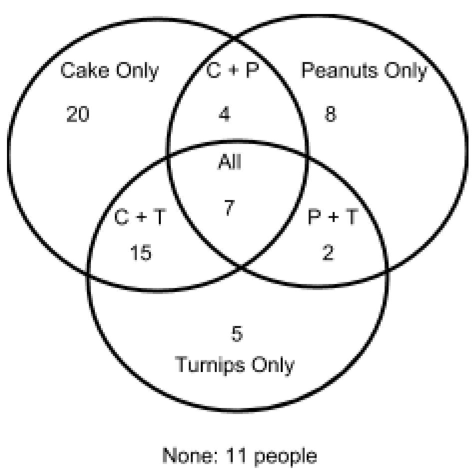
Clearly there is a problem here! As you reconsider your data, you discover that your results actually yielded four groups of people: there are 5 people who like cake only, 11 people who like peanuts only, 14 people who like both cake and peanuts, and 10 people who like neither. It is still true that 19 people like cake and 25 like peanuts, but there is an overlap of 14 people who like both: these people were double counted in the subtraction.

To more clearly visualize the situation above, you may choose to use a Venn Diagram to organize your data:



Given this information, to find the number of people who like neither cake nor peanuts, you can take the total number of people and subtract those who like cake and those who like peanuts, as before, but then you must add back in those who like both, because they were double counted in both cake and peanuts. This gives you 40 - 19 - 25 + 14 = 10, which is the number of people that like neither cake nor peanuts.

The problem becomes a bit more complex if we work with three foods, rather than just two. We add turnips to our informal survey and ask 72 people, and we now have eight groups of respondents: 20 who like cake only, 8 who like peanuts only, 5 who like turnips only, 4 who like cake and peanuts but not turnips, 15 who like cake and turnips but not peanuts, 2 who like turnips and peanuts but not cake, 7 who like all three, and 11 who like none. This Venn Diagram helps to visualize this situation:



Again, we could determine the total number who like none of the three foods by starting with our total and subtracting all who like cake (46), all who like peanuts (21) and all who like turnips (29):

72 - 46 - 21 - 29 = -24

But again, we see that we have double and even triple-counted some respondents, so we need to add back in those who like cake *and* peanuts (11), those who like peanuts *and* turnips (9), and those who like turnips *and* cake (22):

72 - 46 - 21 - 29 + 11 + 9 + 22 = 18

This still isn't correct, because we added back in those who like all three too many times, so we must subtract this group from our total again:

72 - 46 - 21 - 29 + 11 + 9 + 22 - 7 = 11

That is the number that we were expecting!

Solving a problem in this way uses the inclusion-exclusion principle, an important idea in mathematical combinatorics. You can use the same principal to find the number of derangements, and by extension the number of "matches," in any size set (or "stadium," in our problem). However, in order to do this, you need one more key piece of information: combinations.

Combinations (N choose k)

We already know that the total number of possible arrangements of people in stadium seats is N!, if N is the number of seats in the stadium.

But how many possible arrangements are there if the person who was randomly selected to sit in seat number one is the person with seat number one indicated on their ticket? Since we have filled one seat already (in this case, seat one), there are (N-1)! possible seat choices remaining. So the total possible seating arrangements with seat one already filled with the ticket-holder for seat one are (N-1)! seating arrangements. There are also (N-1)! ways to fill the stadium with the person in seat two correctly placed, and (N-1)! ways with the person in seat three correctly placed, and so on.

This may be easier to see if we consider a very small stadium of three people. In a three-seat stadium, we have seats 1, 2, and 3 and the spectators can be arranged in the following ways:

1	***2***	***3***
1	3	2
2	1	***3***
2	3	1
3	1	2
3	***2***	1

As you can see, there are N! total arrangements (3! = 6), and there are (N-1)! arrangements in which seat one is filled by spectator one (2! = 2), (N-1)! arrangements in which seat two is filled by spectator two, and (N-1)! arrangements in which seat three is filled by spectator three.

Because there are (N-1)! arrangements where a spectator is correctly placed in each of the stadium's seats, to get the total number of arrangements where any spectator is seated in their correct seat, we need to multiply (N-1)! by the number of seats in the stadium.

To think about it another way, since the seat in which a person is correctly placed could be any seat in the stadium of N seats, we can use the formula for **N choose 1**, which tells us the number of possible ways that we could choose a single seat from a stadium of size N.

N choose k, the general formula for the situation described above, can be found easily using a graphing calculator, or it can be worked out by hand using the formula:

$$\frac{N!}{k!\,(N-k)!}$$

where N is the number of elements in a set (or seats, in this case), and k is the number of elements that you want to randomly select (in this case, the number of people sitting in their correct seats). When k = 1, this is a pretty easy problem. The number of ways to choose a single seat in a stadium is the same as the number of seats in the stadium, which is the number we expected.

This means that because there are N seats in the stadium, there are (N choose 1)(N-1)! ways that one person could be placed in their correct seat, so to find the number of derangements, we can take N! - (N choose 1)(N-1)!.

However, just like with the cake and peanuts, we know that can't be right! (N choose 1)(N-1)! gives the same answer as N!, and we end up with zero derangements, which we know is not correct. So clearly, as before, we have overcounted something! By subtracting all of the arrangements where any one person is in a correct seat, we have subtracted all of the arrangements with two or more people in the correct seats too many times (such as the first arrangement in the above three-seat stadium example, where all three people are in the correct seats).

To fix this problem, we need to add back in all of the arrangements where at least two people are in correct seats. If there are two people correctly placed, such as the spectators in seat one and seat two, there are (N-2)! ways to seat everyone else. There are **N choose 2** possible ways that we could randomly place two people into correct seats. So to figure out the number of possible arrangements with any two correctly placed people, we multiply (N choose 2)(N-2)!.

Of course, as with the cake, peanuts, and turnips example, adding in all the pairs of correctly-placed spectators is not the end of the process, since we have now added in some arrangements too many times. We need to find out how many arrangements have any three seats correctly filled, or four seats correctly filled, and so on.

To find the number of possible seating arrangements if any three seats are correctly filled, or four, or more, we use the same formula for each additional seat that is correctly filled: (N choose k)(N-k)!

With this background in mind, we can put both of these ideas (the inclusion-exclusion principle and combinations) to work to determine a formula that will give us the number of derangements, and by extension the number of matches, in any size set (stadium).

To determine the number of derangements (like the number of people who liked none of the food choices), start with the total, then subtract all of the matches of at least one seat, add in all of the matches of at least two seats because they were over-subtracted, subtract all of the matches of at least three seats because they were over-added, add in all of the matches of at least four seats because they were over-subtracted, and so on:

N! **-** (N choose 1)(N-1)! **+** (N choose 2)(N - 2)! **-** (N choose 3)(N-3)! **+** (N choose 4)(N-4)! … (N choose N)(N-N)! = The total number of derangements

A match is any stadium in which at least one ticket-holder is sitting in his or her seat, so matches are the opposite of derangements. To find the number of matches, subtract the number of derangements from the total number of arrangements.

Try it with some of your mini-stadiums of 1, 2, 3, 4, or 5. Does this method work? Can you explain why?

Hint: Take another look at the three-seat stadium example. As stated earlier, in the three-seat stadium, there are six arrangements containing at least one correctly-placed seat: two each for seats one, two, and three. Of course, we know that some of these arrangements are really the *same* arrangement. Although it doesn't look like it, there are three arrangements in which two people are correctly placed. Remember that these can be the same arrangement! Can you find all of the correctly placed pairs of spectators?